

# SELF-INTERSECTIONS OF TRAJECTORIES OF LORENTZ PROCESS

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**ABSTRACT.** We study the asymptotic behaviour of the number of self-intersections of a trajectory of a  $\mathbb{Z}^2$ -periodic planar Lorentz process with strictly convex obstacles and finite horizon. We give precise estimates for its expectation and its variance. As a consequence, we establish the almost sure convergence of the self-intersections with a suitable normalization.

## INTRODUCTION

The **Lorentz process** describes the evolution of a point particle moving at unit speed in a domain  $Q$  with elastic reflection on  $\partial Q$ . We consider here a planar Lorentz process in a  $\mathbb{Z}^2$ -periodic domain  $Q \subseteq \mathbb{R}^2$  with strictly convex obstacles  $U_{i,\ell}$  constructed as follows. We choose a finite number of convex open sets  $O_1, \dots, O_I \subset \mathbb{R}^2$  with  $C^3$ -smooth boundary and with non null curvature. We repeat these sets  $\mathbb{Z}^2$ -periodically by defining  $U_{i,\ell} = O_i + \ell$  for every  $(i, \ell) \in \{1, \dots, I\} \times \mathbb{Z}^2$ . We suppose that the closures of the  $U_{i,\ell}$  are pairwise disjoint. Now we define the domain  $Q := \mathbb{R}^2 \setminus \bigcup_{i=1}^I \bigcup_{\ell \in \mathbb{Z}^2} U_{i,\ell}$ . We assume that **the horizon is finite**, which means that every line meets the boundary of  $Q$  (i.e. there is no infinite free flight). We consider a point particle moving in  $Q$  with unit speed and with respect to the Descartes reflection law at its reflection times (reflected angle=incident angle). We call configuration of a particle at some time the couple constituted by its position and its speed. The Lorentz process in the domain  $Q$  is the flow  $(Y_t)_t$  on  $Q \times S^1$  such that  $Y_t$  maps the configuration at time 0 to the configuration at time  $t$ . We assume that the initial distribution  $\mathbb{P}$  is uniform on  $(Q \cap [0, 1]^2) \times S^1$ . The study of the Lorentz process is strongly related to the corresponding Sinai billiard  $(\bar{M}, \bar{\mu}, \bar{T})$ . Recall that this billiard is the probability dynamical system describing the dynamics of the Lorentz process modulo  $\mathbb{Z}^2$  and at reflection times. Ergodic properties of this dynamical system have been studied namely by Sinai in [19] (for its ergodicity), Bunimovich and Sinai [2, 3], Bunimovich, Chernov and Sinai [4, 5] (for central limit theorems), Young [21] (for exponential rate of decorrelation). Other limit theorems for the Sinai billiard and its applications to the Lorentz process have been investigated in many papers, let us mention namely [7, 14, 20] for its ergodicity and [9] for some other properties.

We are interested here in the study of the following quantity, called **number of self-intersections of the trajectory of the Lorentz Process**:

$$\mathcal{V}_t := \#\{(r, s) \in [0; t]^2 : \pi_Q(Y_r) = \pi_Q(Y_s)\},$$

where  $\pi_Q$  denotes the canonical projection from  $Q \times S^1$  to  $Q$  (i.e.  $\pi_Q(q, \vec{v}) = q$ ). This quantity  $\mathcal{V}_t$  corresponds to the number of couples of times  $(r, s)$  before time  $t$  such that the particle was at the same position in the plane at both times  $r$  and  $s$ . We also define  $V_n$  as the number of self-intersections up to the  $n$ th reflection time. The studies of  $\mathcal{V}_t$  and of  $V_n$  are naturally linked.

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Self-intersections of random walks have been studied by many authors (see [6] and references therein). Motivated by the study of planar random walks in random sceneries, Bolthausen [1] established an exact estimate for the expectation of the number of self-intersections of planar recurrent random walks. He also stated an upper bound for its variance. This last estimate was sufficient for his purpose but not optimal. A precise estimate for this variance has recently been stated by Deligiannidis and Utev [8].

In view of planar Lorentz process in random scenery, another notion of self-intersections of Lorentz process arises: the number of self-intersections of the Lorentz process seen on obstacles, i.e. the number  $\hat{V}_n$  of couples of times  $(r, s)$  (before the  $n$ -th reflection) such that the particle hit the same obstacle at both times  $r$  and  $s$ . This quantity has been studied in [16, 17]. In the present work, our approach has some common points with [16, 17] but the study of  $V_n$  (and thus of  $\mathcal{V}_t$ ) is much more delicate than the study of  $\hat{V}_n$  (see section 2 for some explanations).

Let us define  $(\mathcal{I}_k, S_k)$  in  $\{1, \dots, I\} \times \mathbb{Z}^2$  as the index of the obstacle hit at the  $k$ -th reflection time ( $(I_0, S_0)$  being the index of the obstacle at time 0 or at the last reflection time before 0). The asymptotic behaviour of  $(S_n)_n$  plays some role here. In particular, our proofs use a decorrelation result and some precised local limit theorems for  $(S_n)_n$ . As a consequence, the constants appearing in our statements are expressed in terms of the asymptotic (positive) variance matrix  $\Sigma^2$  of  $(k^{-1/2}S_k)_{k \geq 1}$  (with respect to  $\bar{\mu}$ ).

**Theorem 1.** *We have*

$$\mathbb{E}_{\bar{\mu}}[V_n] = cn \log n + O(n), \quad \text{with } c := 2\mathbb{E}_{\bar{\mu}}[\tau]/(\pi\sqrt{\det \Sigma^2} \sum_i |\partial O_i|), \quad (1)$$

where  $\tau$  is the free flight length until the next reflection time.

**Theorem 2.** *We have*

$$\mathbb{E}(\mathcal{V}_t) = \frac{2t \log t}{\pi\sqrt{\det \Sigma^2} \sum_i |\partial O_i|} + O(t) \quad \text{as } t \text{ goes to infinity.} \quad (2)$$

Let us indicate that these results are generalized in Corollaries 15 and 17 to a wider class of initial probability measures.

**Theorem 3.** *We have  $\text{Var}_{\bar{\mu}}(V_n) \sim c'n^2$  with*

$$c' := c^2 \left(1 + 2J - \frac{\pi^2}{6}\right) \quad \text{and } J := \int_{[0,1]^3} \frac{(1 - (u + v + w))\mathbf{1}_{\{u+v+w \leq 1\}} du dv dw}{uv + uw + vw}.$$

**Corollary 4.** *The following convergences hold almost everywhere (with respect to  $\bar{\mu}$  and to the Lebesgue measure on  $Q \times S^1$  respectively):*

$$\lim_{n \rightarrow +\infty} \frac{V_n}{n \log n} = c \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{\mathcal{V}_t}{t \log t} = \frac{2}{\pi\sqrt{\det \Sigma^2} \sum_i |\partial O_i|}.$$

The paper is organized as follows. In Section 1, we introduce the billiard systems, some notations and local limit theorems with remainder terms. In Section 2, we prove Theorem 1. In Section 3, we establish a decorrelation result in view of our proof of Theorem 3 in Section 4. In Section 5, we use Theorems 1 and 3 to prove Theorem 2 and some generalization of Theorems 1 and 2 to a class of probability measures. Finally we prove Corollary 4 in Section 6.

## 1. LORENTZ PROCESS AND BILLIARD SYSTEMS

We denote by  $\langle \cdot, \cdot \rangle$  the usual scalar product on  $\mathbb{R}^2$  and by  $|\cdot|$  the supremum norm on  $\mathbb{R}^2$ .

**1.1. planar billiard system.** For any  $q \in \partial Q$ , we write  $\vec{n}_q$  for the unit normal vector to  $\partial Q$  at  $q$  directed into  $Q$ . We consider the set  $M$  of couples position-unit speed  $(q, \vec{v})$  corresponding to a reflected vector on  $\partial Q$ :

$$M := \{(q, \vec{v}) \in (\partial Q) \times S^1 : \langle \vec{n}_q, \vec{v} \rangle \geq 0\}.$$

For every  $i \in \{1, \dots, I\}$ , we fix some  $q_i \in \partial O_i$ . A couple  $(q, \vec{v}) \in M$  is parametrized by  $(i, \ell, r, \varphi) \in \bigcup_{i'=1}^I \{i'\} \times \mathbb{Z}^2 \cup \frac{\mathbb{R}}{|\partial O_{i'}| \mathbb{Z}} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$  if

- $q - \ell$  is the point of  $\partial O_i$  with curvilinear absciss  $r$  for the trigonometric orientation (starting from  $q_i$ )
- $\varphi$  is the angular measure of  $\widehat{(\vec{n}_q, \vec{v})}$ .

We consider the transformation  $T$  mapping a reflected vector to the reflected vector corresponding to the next collision time.  $T$  preserves the (infinite) measure  $\mu$  with density  $\cos(\varphi)$  with respect to the measure  $drd\varphi$  on  $M$ . This infinite measure dynamical system  $(M, \mu, T)$  is called **planar billiard system**. We endow  $M$  with a metric  $d$  equal to  $\max(|r - r'|, |\varphi - \varphi'|)$  on any obstacle  $\partial U_{i,\ell}$ . We define the map  $\tau : M \rightarrow [0, +\infty[$  by

$$\tau(q, \vec{v}) := \min\{s > 0 : q + s\vec{v} \in \partial Q\},$$

which corresponds to the length of the free flight of a particle starting from  $q$  with initial speed  $\vec{v}$ . Due to our assumptions, we have

$$\min \tau > 0 \quad \text{and} \quad \max \tau < \infty.$$

We define  $R_0$  as the set of  $(q, \vec{v}) \in M$  with  $\vec{v}$  tangent to  $\partial Q$  at  $q$  (this set corresponds to  $\{\varphi = 0\}$ ). For any integers  $k \leq \ell$ , we write  $R_{k,\ell} = \bigcap_{m=k}^{\ell} T^m(R_0)$  and  $\xi_k^\ell$  for the set of connected components of  $M \setminus R_{-\ell, -k}$ . Due to the hyperbolic properties of  $T$ , it is easy to see that (see for example [18, Lemma A.1])

$$\exists c_0 > 0, \exists \delta \in (0, 1), \forall k \geq 1, \forall \mathcal{C} \in \xi_{-k}^k, \text{diam}(\mathcal{C}) \leq c_0 \delta^k. \quad (3)$$

We recall that  $T$  is discontinuous but  $\frac{1}{2}$ -Hölder continuous on each connected component of  $M \setminus R_{-1,0}$ .

**1.2. Lorentz process.** To avoid ambiguity, at collision times, we only consider reflected vectors. The set of configurations is then

$$\mathcal{M} := ((Q \setminus \partial Q) \times S^1) \cup M \subseteq Q \times S^1.$$

The Lorentz process is the flow  $(Y_t)_t$  defined on  $\mathcal{M}$  such that, for every  $(q, \vec{v}) \in \mathcal{M}$ ,  $Y_t(q, \vec{v}) = (q_t, \vec{v}_t)$  is the couple position-speed at time  $t$  of a particle that was at position  $q$  with speed  $\vec{v}$  at time 0. This flow preserves the measure  $\nu$  on  $\mathcal{M}$ , where  $\nu$  is the product of the Lebesgue measure on  $Q$  and of the uniform measure on  $S^1$ .

This flow is naturally identified with the suspension flow  $(\tilde{Y}_t)_t$  over  $(M, \mu, T)$  with roof function  $\tau$ . Indeed, we recall that  $(\tilde{Y}_t)_t$  is defined by  $\tilde{Y}_t(x, s) = (x, s + t)$  on the set

$$\tilde{\mathcal{M}} := \{(x, s) \in M \times [0, +\infty[ : s \leq \tau(x)\}, \text{ with the identifications } (x, \tau(x)) \equiv (T(x), 0).$$

The flow  $(\tilde{Y}_t)_t$  preserves the measure  $\tilde{\nu}$  on  $\tilde{\mathcal{M}}$  given by  $d\tilde{\nu}(x, s) = d\mu(x)ds$ . Now, we define  $\Delta : \tilde{\mathcal{M}} \rightarrow \mathcal{M}$  by

$$\Delta((q, \vec{v}), s) = (q + s\vec{v}, \vec{v}) \quad \text{if } s < \tau(q, \vec{v}).$$

We have

$$Y_t = \Delta \circ \tilde{Y}_t \circ \Delta^{-1} \quad \text{and} \quad \Delta_*(\tilde{\nu}) = \nu. \quad (4)$$

**1.3. Billiard system with finite measure.** We define  $\bar{M}$  as the set of  $(q, \vec{v}) \in M$  such that  $q \in \bigcup_{i=1}^I \partial O_i$ . A point of  $\bar{M}$  is now parametrized by  $(i, r, \varphi)$ . We consider the transformation  $\bar{T} : \bar{M} \rightarrow \bar{M}$ , corresponding to  $T$  modulo  $\mathbb{Z}^2$ . More precisely, if  $T(q, \vec{v}) = (q', \vec{v}')$ , then  $\bar{T}(q, \vec{v}) = (q'', \vec{v})$  with  $q'' \in (q' + \mathbb{Z}^2) \cap \bigcup_{i=1}^I \partial O_i$ . This transformation  $\bar{T}$  preserves the probability measure  $\bar{\mu}$  of density  $\cos(\varphi)/(2 \sum_i |\partial O_i|)$  with respect to  $dr d\varphi$ .

We call **toral billiard system** the probability dynamical system  $(\bar{M}, \bar{\mu}, \bar{T})$ .

It is easy to see that  $(M, \mu, T)$  corresponds to the cylindrical extension of  $(\bar{M}, \bar{\mu}, \bar{T})$  by  $\Psi : \bar{M} \rightarrow \mathbb{Z}^2$  given by  $\Psi = (S_1)_{|\bar{M}}$  (with  $S_n$  defined in the introduction). Indeed

$$\forall ((q, \vec{v}), \ell) \in \bar{M} \times \mathbb{Z}^2, \quad T(q + \ell, \vec{v}) = (q' + \ell + \Psi(q, \vec{v}), \vec{v}') \quad \text{if } (q', \vec{v}') = \bar{T}(q, \vec{v}).$$

More generally we have

$$\forall ((q, \vec{v}), \ell) \in \bar{M} \times \mathbb{Z}^2, \quad \forall n \geq 1, \quad T^n(q + \ell, \vec{v}) = (q_n + \ell + \sum_{k=0}^{n-1} \Psi(\bar{T}^k(q, \vec{v})), \vec{v}_n) \quad \text{if } (q_n, \vec{v}_n) = \bar{T}^n(q, \vec{v}). \quad (5)$$

Observe that  $\sum_{k=0}^{n-1} \Psi \circ \bar{T}^k = S_n$  on  $\bar{M}$ . We recall the following local limit theorem with remainder term. We set  $\beta := \frac{1}{2\pi\sqrt{\det \Sigma^2}}$ .

**Proposition 5** (Proposition 4.1 of [18]). *Let  $p > 1$ . There exists  $c > 0$  such that, for any  $k \geq 1$ , if  $A \subseteq \bar{M}$  is a union of components of  $\xi_{-k}^k$  and  $B \subseteq \bar{M}$  is a union of components  $\xi_{-k}^\infty$ , then for any  $n > 2k$  and  $N \in \mathbb{Z}^2$*

$$\left| \bar{\mu}(A \cap \{S_n = N\} \cap \bar{T}^{-n}(B)) - \frac{\beta e^{-\frac{1}{2(n-2k)} \langle (\Sigma^2)^{-1} N, N \rangle}}{n - 2k} \bar{\mu}(A) \bar{\mu}(B) \right| \leq \frac{ck \bar{\mu}(B)^{\frac{1}{p}}}{(n - 2k)^{\frac{3}{2}}}.$$

Note that, if we suppose  $n \geq 3k$ , we can replace the conclusion of this result by

$$\left| \bar{\mu}(A \cap \{S_n = N\} \cap \bar{T}^{-n}(B)) - \frac{\beta e^{-\frac{1}{2n} \langle (\Sigma^2)^{-1} N, N \rangle}}{n} \bar{\mu}(A) \bar{\mu}(B) \right| \leq \frac{ck \bar{\mu}(B)^{\frac{1}{p}}}{n^{\frac{3}{2}}}. \quad (6)$$

**Remark 6.** *Observe that, since the billiard system  $(\bar{M}, \bar{\mu}, \bar{T})$  is time reversible, if  $A \subseteq \bar{M}$  is a union of components of  $\xi_{-\infty}^k$  and  $B \subseteq \bar{M}$  is a union of components  $\xi_{-k}^k$ , if  $n > 3k$  then we have*

$$\left| \bar{\mu}(A \cap \{S_n = N\} \cap \bar{T}^{-n}(B)) - \frac{\beta e^{-\frac{1}{2n} \langle (\Sigma^2)^{-1} N, N \rangle}}{n} \bar{\mu}(A) \bar{\mu}(B) \right| \leq \frac{ck \bar{\mu}(A)^{\frac{1}{p}}}{n^{\frac{3}{2}}}. \quad (7)$$

Estimates (6) and (7) will be enough most of the time but not every time. We will also use the following refinements of the local limit theorem.

**Proposition 7** (Proposition 4 of [16]). *Let any real number  $p > 1$ . There exist  $a_0 > 0$  and  $K_1 > 0$  such that, for any integers  $k \geq 0$ ,  $n \geq 1$ , any measurable set  $A \subseteq \bar{M}$  union of elements of  $\xi_0^k$ , any measurable set  $B \subseteq \bar{M}$  union of elements of  $\xi_0^{+\infty}$ , for any  $N \in \mathbb{Z}^2$ , we have*

$$\begin{aligned} & \left| \bar{\mu}(A \cap \{S_{n+k} - S_k = N\} \cap \bar{T}^{-(n+k)} B) - \frac{\beta \bar{\mu}(A) \bar{\mu}(B)}{n} e^{-\frac{1}{2n} \langle (\Sigma^2)^{-1} N, N \rangle} \right| \\ & \leq K_1 \left( \frac{\bar{\mu}(B) + \bar{\mu}(A) \bar{\mu}(B)^{\frac{1}{p}}}{n^{\frac{3}{2}}} \left( \frac{|N|}{\sqrt{n}} + \frac{|N|^3}{n^{\frac{3}{2}}} \right) e^{-\frac{a_0}{n} |N|^2} + \frac{\bar{\mu}(B)^{\frac{1}{p}}}{n^2} \right). \end{aligned}$$

We generalize this result as follows.

**Proposition 8.** *Let any real number  $p > 1$ . There exist  $C > 0$ ,  $a_0 > 0$  and  $K_1 > 0$  such that, for any integers  $k \geq 0$ ,  $n \geq 1$  such that  $n \geq 4k$ , any measurable set  $A \subseteq \bar{M}$  union of elements of  $\xi_{-k}^k$ , any measurable set  $B \subseteq \bar{M}$  union of elements of  $\xi_{-k}^{+\infty}$ , for any  $N \in \mathbb{Z}^2$ , we have*

$$\begin{aligned} & \left| \bar{\mu}(A \cap \{S_n = N\} \cap \bar{T}^{-n}B) - \frac{\beta \bar{\mu}(A) \bar{\mu}(B)}{n} e^{-\frac{1}{2n} \langle (\Sigma^2)^{-1} N, N \rangle} \right| \\ & \leq K_1 k \left( \frac{\bar{\mu}(B) + \bar{\mu}(A) \bar{\mu}(B)^{\frac{1}{p}}}{n^{\frac{3}{2}}} \left( \frac{|N|}{\sqrt{n}} + \frac{|N|^3}{n^{\frac{3}{2}}} \right) e^{-\frac{a_0}{n} (\max(|N|-2k, 0))^2} + k \frac{\bar{\mu}(B)^{\frac{1}{p}}}{n^2} \right). \end{aligned}$$

*Proof.* Observe that  $\bar{T}^{-k}A$  is a union of elements of  $\xi_0^{2k}$  and that  $\bar{T}^{-k}B$  is a union of elements of  $\xi_0^{+\infty}$ . We have

$$\begin{aligned} \bar{\mu}(A \cap \{S_n = N\} \cap \bar{T}^{-n}B) &= \bar{\mu}(\bar{T}^{-k}A \cap \{S_{n+k} - S_k = N\} \cap \bar{T}^{-(n+k)}B) \\ &= \sum_{x', y'} \bar{\mu}(A_{x'}; S_n - S_{2k} = N - x' - y'; \bar{T}^{-n}B_{y'}), \end{aligned}$$

with  $A_{x'} := \bar{T}^{-k}A \cap \{S_{2k} - S_k = x'\}$  and  $B_{y'} := \bar{T}^{-k}B \cap \{S_k = y'\}$  and where the sum is taken over  $x', y' \in \mathbb{Z}^2$  such that  $|x'| \leq k \|S_1\|_\infty$  and  $|y'| \leq k \|S_1\|_\infty$ ; Applying Proposition 7 with  $(A_{x'}, B_{y'})$  and using the fact that  $n - 2k \geq n/2$ , we obtain the result.  $\square$

**Remark 9.** *Observe again that, by time reversibility, if  $A$  is a union of elements of  $\xi_{-\infty}^k$ , if  $B$  is a union of components of  $\xi_{-k}^k$  and if  $n \geq 4k$ , then*

$$\begin{aligned} & \left| \bar{\mu}(A \cap \{S_n = N\} \cap \bar{T}^{-n}B) - \frac{\beta \bar{\mu}(A) \bar{\mu}(B)}{n} e^{-\frac{1}{2n} \langle (\Sigma^2)^{-1} N, N \rangle} \right| \\ & \leq K_1 k \left( \frac{\bar{\mu}(A) + \bar{\mu}(B) \bar{\mu}(A)^{\frac{1}{p}}}{n^{\frac{3}{2}}} \left( \frac{|N|}{\sqrt{n}} + \frac{|N|^3}{n^{\frac{3}{2}}} \right) e^{-\frac{a_0}{n} (\max(|N|-2k, 0))^2} + k \frac{\bar{\mu}(A)^{\frac{1}{p}}}{n^2} \right). \end{aligned}$$

## 2. PROOF OF THEOREM 1

Observe that the trajectory of the particle (starting from  $M$ ) up to the  $n$ -th reflection is  $\bigcup_{j=0}^{n-1} [\pi_Q \circ T^j, \pi_Q \circ T^{j+1}]$ . So we have  $\bar{\mu}$ -almost surely

$$V_n = \sum_{k,j=0}^{n-1} \mathbf{1}_{E_{k,j}} = n + 2 \sum_{k=1}^{n-1} \sum_{j=0}^{n-1-k} \mathbf{1}_{E_{j,j+k}},$$

with

$$E_{j,k} := \{[\pi_Q \circ T^j, \pi_Q \circ T^{j+1}] \cap [\pi_Q \circ T^k, \pi_Q \circ T^{k+1}] \neq \emptyset\}.$$

Hence

$$\bar{\mu}(V_n) = n + 2 \sum_{k=1}^n (n-k) \bar{\mu}(E_{0,k}). \quad (8)$$

**Proposition 10.** *There exists  $\eta > 0$  such that  $\bar{\mu}(E_{0,n}) = c/(2n) + O(n^{-1-\eta})$ , with  $c$  defined in (1).*

*Proof of Theorem 1.* It follows directly from (8) and from Proposition 10. Indeed

$$\sum_{k=1}^n \frac{n-k}{k} = n \left( \sum_{k=1}^n k^{-1} \right) - n = n \log(n) + O(n)$$

and

$$\sum_{k=1}^n \frac{n-k}{k^{1+\eta}} = n \left( \sum_{k=1}^n k^{-1-\eta} \right) - \sum_{k=1}^n k^{-\eta} = O(n).$$

□

Before going into the proof of Proposition 10, let us see the common points between  $\hat{V}_n$  and  $V_n$  and let us also explain why the study of  $V_n$  requires more subtle estimates than the study of  $\hat{V}_n$ . Recall that  $\hat{V}_n = \sum_{k,j=1}^n \mathbf{1}_{(\mathcal{I}_k, S_k) = (\mathcal{I}_j, S_j)}$ . So  $\mathbb{E}_{\bar{\mu}}[\hat{V}_n] = n + 2 \sum_{k=1}^{n-1} (n-k) \bar{\mu}(\hat{E}_{0,k})$ , with  $\hat{E}_{0,k} = \bigcup_{i=1}^I \{I_0 = i, S_k = 0, \mathcal{I}_k = i\}$ . This expression may appear similar to (8), but  $E_{0,k}$  is more complicate than  $\hat{E}_{0,k}$ . Indeed, in  $\bar{M}$ , we have

$$\begin{aligned} E_{0,k} &= \bigcup_{x \in \bar{M}} (\{x\} \cap T^{-k}(V^{(x)})) \\ &= \bigcup_{N \in \mathbb{Z}^2} \bigcup_{x \in \bar{M}} (\{x\} \cap \{S_k = N\} \cap \bar{T}^{-k}(\bar{M} \cap (V^{(x)} - N))), \quad \text{due to (5)} \end{aligned}$$

with

$$V^{(x)} := \{y \in M : [\pi_Q(y), \pi_Q \circ T(y)] \cap [\pi_Q(x), \pi_Q \circ T(x)] \neq \emptyset\}$$

and with  $A - N = \{(q - N, \vec{v}) : (q, \vec{v}) \in A\}$ . The union on  $N$  is not a problem (it is a finite union since the horizon is finite), the main problem is that the union on  $x$  is not finite. Indeed the set  $V^{(x)}$  depends on  $x$  (and not only on the obstacle containing  $x$ ).

**Lemma 11.** *We have  $\bar{\mu}(V^{(x)} + \mathbb{Z}^2) = \frac{2\tau(x)}{\sum_i |\partial O_i|}$ .*

*Proof.* We use the fact that the measure  $\cos \varphi \, dr d\varphi$  is preserved by billiard maps. So, adding the virtual obstacle  $[\pi_Q(x), \pi_Q(T(x))]$ , we obtain that  $\mu(V^{(x)})$  is equal to the measure of the set of vectors based on  $[\pi_Q(x), \pi_Q \circ T(x)]$  for the measure  $|\cos \varphi| \, dr d\varphi$ , which is equal to  $4\tau(x)$  (since  $\tau(x)$  is the length of  $[\pi_Q(x), \pi_Q(T(x))]$ ). □

*Proof of Proposition 10.* There exists  $C > 0$  such that, for any  $\varepsilon > 0$ , any integer  $n \geq 1$ , any  $x_0 \in \bar{M}$ , any connected component  $\mathcal{C}$  of  $B(x_0, \varepsilon) \setminus R_{-1,0}$  and any  $x \in \mathcal{C}$ , we have

$$(\mathcal{C} \cap E_{0,n}) \Delta (\mathcal{C} \cap T^{-n}V^{(x)}) \subseteq \mathcal{C} \cap T^{-n}\mathcal{D}_{\mathcal{C}},$$

with

$$\mathcal{D}_{\mathcal{C}} := \pi_Q^{-1}\pi_Q(\mathcal{C}) \cup (\pi_Q^{-1}\pi_Q(T(\mathcal{C})) \cup T^{-1}(\pi_Q^{-1}\pi_Q(\mathcal{C})) \cup T^{-1}(\pi_Q^{-1}\pi_Q(T(\mathcal{C}))) \subseteq \mathcal{E}_{x,\varepsilon}$$

and

$$\mathcal{E}_{x,\varepsilon} := \pi_Q^{-1}\pi_Q(B(x, \varepsilon) \cup B(T(x), C\sqrt{\varepsilon})) \cup T^{-1}(\pi_Q^{-1}\pi_Q(B(x, \varepsilon) \cup B(T(x), C\sqrt{\varepsilon}))),$$

since  $T$  is  $\frac{1}{2}$ -Hölder continuous on each connected component of  $M \setminus R_{-1,0}$ . Take  $(\varepsilon, k)$  such that  $\varepsilon^2 = n^{-\frac{1}{10}} = \delta^k$  (with  $\delta$  of (3)). For any connected component  $\mathcal{C}$  of  $B(x_0, \varepsilon) \setminus R_{-1,0}$ , we choose (in a measurable way) a point  $x = x_{\mathcal{C}} \in \mathcal{C}$  and define

$$\tilde{E}_{n,\mathcal{C}} := \tilde{\mathcal{C}} \cap T^{-n}\tilde{V}^{(x)}, \quad \text{with } \tilde{\mathcal{C}} := \bigcup_{Z \in \xi_{-k}^k: Z \cap \mathcal{C} \neq \emptyset} Z \quad \text{and} \quad \tilde{V}^{(x)} := \bigcup_{Z \in \xi_{-k}^k: Z \cap V^{(x)} \neq \emptyset} Z. \quad (9)$$

We have  $|\bar{\mu}(\mathcal{C} \cap E_{0,n}) - \bar{\mu}(\tilde{E}_{n,\mathcal{C}})| \leq \bar{\mu}(\tilde{\mathcal{D}}_{n,\mathcal{C}})$ , with

$$\tilde{\mathcal{D}}_{n,\mathcal{C}} := \tilde{\mathcal{C}} \cap T^{-n}\tilde{\mathcal{D}}_{\mathcal{C}}, \quad \text{with } \tilde{\mathcal{D}}_{\mathcal{C}} := \bigcup_{Z \in \xi_{-k}^k: Z \cap \mathcal{D}_{\mathcal{C}} \neq \emptyset} Z. \quad (10)$$

Observe that  $\pi_Q(\bigcup_{x \in \bar{M}} V^{(x)})$  is contained in  $\bigcup_{i=1}^I \bigcup_{|\ell| \leq \|S_1\|_\infty} (\partial O_i + \ell)$ . Therefore, due to (5)

$$\tilde{E}_{n,\mathcal{C}} = \bigcup_{|\ell| \leq \|S_1\|_\infty} (\tilde{\mathcal{C}} \cap \{S_n = \ell\} \cap \bar{T}^{-n}(\bar{M} \cap (\tilde{V}^{(x)} - \ell))) \quad (11)$$

and

$$\tilde{\mathcal{D}}_{n,\mathcal{C}} = \bigcup_{|\ell| \leq \|S_1\|_\infty} (\tilde{\mathcal{C}} \cap \{S_n = \ell\} \cap \bar{T}^{-n}(\bar{M} \cap (\tilde{\mathcal{D}}_{\mathcal{C}} - \ell))). \quad (12)$$

Let  $p \in (1, 2)$ . Due to (7) and (3), we conclude that there exist  $\tilde{C}, \tilde{C}_0, \tilde{C}_1 > 0$  such that, for any  $\varepsilon > 0$ , any integer  $n \geq 1$ , any  $x_0 \in M$ , any connected component  $\mathcal{C}$  of  $B(x_0, \varepsilon) \setminus R_{-1,0}$  and any  $x \in \mathcal{C}$ , we have

$$|\bar{\mu}(\mathcal{C} \cap E_{0,n}) - \bar{\mu}(\tilde{E}_{n,\mathcal{C}})| \leq \bar{\mu}(\tilde{\mathcal{D}}_{n,\mathcal{C}}) \leq \tilde{C} \left( \frac{\varepsilon^2 \delta^k}{n} + \frac{k \varepsilon^{\frac{2}{p}}}{n^{\frac{3}{2}}} \right) \leq \tilde{C}_0 \frac{\varepsilon^2 \delta^k}{n}$$

and

$$\begin{aligned} \bar{\mu}(\mathcal{C} \cap E_{0,n}) &= \pm \bar{\mu}(\tilde{\mathcal{D}}_{n,\mathcal{C}}) + \bar{\mu}(\tilde{E}_{n,\mathcal{C}}) \\ &= \pm \tilde{c} \left( \frac{\varepsilon^2 \delta^k}{n} + \frac{k \varepsilon^{\frac{2}{p}}}{n^{\frac{3}{2}}} \right) + \frac{2\beta \bar{\mu}(\mathcal{C}) \bar{\mu}(V^{(x)}) (1 \pm \delta^{\frac{k}{2}})}{n} \\ &= \pm 2\tilde{c} n^{-\frac{23}{20}} + \frac{2\beta \bar{\mu}(\mathcal{C}) \tau(x)}{n \sum_i |\partial O_i|}. \end{aligned}$$

Let  $m \geq 1$ . We consider a  $\bar{\mu}$ -essential partition of  $\bar{M}$  in rectangles  $(P_m^{(i,j,\ell)})_{i \in \{1, \dots, I\}, j, \ell \in \{0, \dots, m-1\}}$  given by

$$P_m^{(i,j,\ell)} := \left\{ (i, \bar{r}, \varphi) : r \in \left[ \frac{j|\partial O_i|}{m}; \frac{(j+1)|\partial O_i|}{m} \right], \varphi \in \left[ -\frac{\pi}{2} + \frac{\ell\pi}{m}; -\frac{\pi}{2} + \frac{(\ell+1)\pi}{m} \right] \right\}.$$

We write  $\mathcal{P}_m$  for the union on  $(i, j, \ell)$  of the partition of  $P_m^{(i,j,\ell)} \setminus R_{-1,0}$  in connected components. Taking  $\varepsilon^{-1} = m = n^{1/20}$  and  $k$  such that  $\delta^k = n^{-1/10}$ . We obtain

$$\begin{aligned} \bar{\mu}(E_{0,n}) &= \sum_{\mathcal{C} \in \mathcal{P}_m} \bar{\mu}(\mathcal{C} \cap E_{0,n}) \\ &= \pm n^{-\frac{21}{20}} + \sum_{\mathcal{C} \in \mathcal{P}_m} \frac{2\beta \mathbb{E}_{\bar{\mu}}[\tau \mathbf{1}_{\mathcal{C}}]}{n \sum_i |\partial O_i|} \\ &= \frac{2\beta \mathbb{E}_{\bar{\mu}}[\tau]}{n \sum_i |\partial O_i|} + O(n^{-\frac{21}{20}}), \end{aligned}$$

using the fact that  $\tau$  is  $1/2$ -Hölder continuous on each connected component of  $\bar{M} \setminus R_{-1,0}$ .  $\square$

### 3. A DECORRELATION RESULT

Let us recall some facts on the towers constructed by Young [21]. These towers are two dynamical systems  $(\tilde{M}, \tilde{\mu}, \tilde{T})$  and  $(\hat{M}, \hat{\mu}, \hat{T})$  such that  $(\tilde{M}, \tilde{\mu}, \tilde{T})$  is an extension of  $(\hat{M}, \hat{\mu}, \hat{T})$  and  $(\bar{M}, \bar{\mu}, \bar{T})$ . This means that there exist two measurable maps  $\tilde{\pi} : \tilde{M} \rightarrow \bar{M}$  and  $\hat{\pi} : \hat{M} \rightarrow \bar{M}$  such that:  $\tilde{\pi} \circ \tilde{T} = \bar{T} \circ \tilde{\pi}$ ,  $\hat{\pi} \circ \hat{T} = \bar{T} \circ \hat{\pi}$ ,  $\bar{\mu} = (\tilde{\pi})_* \tilde{\mu}$  and  $\hat{\mu} = (\hat{\pi})_* \tilde{\mu}$ . Young defines a separation time  $\hat{s}$  on  $\hat{M}$  such that if  $\hat{s}(x, y) \geq n$ , we have  $\hat{s}(x, y) = n + \hat{s}(\hat{T}^n x, \hat{T}^n y)$  and  $\tilde{\pi} \hat{\pi}^{-1}(\{x\})$ ,  $\tilde{\pi} \hat{\pi}^{-1}(\{y\})$  are contained in the same atom of  $\xi_0^n$ . For any  $\beta_0 \in (0, 1)$  and any  $\varepsilon_0 \geq 0$ , Young defines a Banach space  $(\mathcal{V}_{\beta_0, \varepsilon_0}, \|\cdot\|_{(\beta_0, \varepsilon_0)})$  containing  $\mathbf{1}_{\hat{M}}$ . Let  $p$  be fixed and set  $q := p/(p-1)$ . It is possible to find  $\beta_0 \in (0, 1)$  and  $\varepsilon_0 > 0$  such that

$$\|\cdot\|_{L^q(\hat{\mu})} \leq C_0 \|\cdot\|_{(\beta_0, \varepsilon_0)}, \quad \text{for some } C_0 > 0. \quad (13)$$

From now on, we write  $(\mathcal{V}, \|\cdot\|) = (\mathcal{V}_{\beta_0, \varepsilon_0}, \|\cdot\|_{(\beta_0, \varepsilon_0)})$  for this choice of  $(\beta_0, \varepsilon_0)$ . Lemma 10 of [16] states that

$$\|gh\| \leq \|g\|_{(\beta_0, 0)} \|h\|. \quad (14)$$

We recall that, due to Young's construction, if  $f$  is constant on each element of  $\xi_0^N$ , then there exists a measurable  $\hat{f}$  defined on  $\hat{M}$  such that

$$f \circ \tilde{\pi} = \hat{f} \circ \hat{\pi} \quad \text{with} \quad \|\hat{f}\|_{(\beta_0, 0)} \leq \|f\|_\infty (1 + 2\beta_0^{-N}). \quad (15)$$

Let  $P$  be the transfer operator on  $L^q$  of  $f \mapsto f \circ \hat{T}$  seen as an operator on  $L^p$ . Young proved the quasicompacity of this operator  $P$  on  $\mathcal{V}$ . As in [16], we consider here an adaptation of the construction of Young's towers such that 1 is the only dominating eigenvalue of  $P$  on  $\mathcal{V}$  and has multiplicity one. Hence, there exist  $K_0 > 0$  and  $a > 0$  such that

$$\forall n \geq 1, \quad \|P^n(\cdot) - \mathbb{E}_{\hat{\mu}}[\cdot]\| \leq K_0 e^{-an}. \quad (16)$$

Thanks to this property, Young established an exponential rate of decorrelation. Let us consider  $\Psi : \bar{M} \rightarrow \mathbb{Z}^2$  the cell-shift function. Recall that, on  $\bar{M}$ ,  $S_n = \sum_{k=0}^{n-1} \Psi \circ \bar{T}^k$ . Since  $\Psi$  is constant on each element of  $\xi_0^1$ , there exists  $\hat{\Psi} : \hat{M} \rightarrow \mathbb{Z}^2$  such that  $\hat{\Psi} \circ \hat{\pi} = \Psi \circ \pi$  and the coordinates of  $\hat{\Psi}$  are in  $\mathcal{V}_{(\beta_0, 0)}$  with norm less than  $3\beta_0^{-1} \|\Psi\|_\infty$ . For any  $u \in \mathbb{R}^2$ , we define  $P_u(f) = P(e^{i\langle u, \hat{\Psi} \rangle} f)$ . Observe that

$$\forall k \geq 1, \quad P_u^k(f) = P^k(e^{i\langle u, \hat{S}_k \rangle} f) \quad \text{and} \quad P_u^k(f \circ \hat{T}^k \times g) = f P_u^k(g), \quad (17)$$

with  $\hat{S}_n := \sum_{k=0}^{n-1} \hat{\Psi} \circ \hat{T}^k$ . In [20], Szász and Varjú applied the classical Nagaev-Guivarc'h method [12, 13, 11] to this context. This method plays a crucial role in the proof of Proposition 12 and gives in particular the following inequalities (see [20] and Lemma 12 of [16])

$$K_1 := \sup_{u \in [-\pi, \pi^2]} \|P_u^k\| < \infty, \quad (18)$$

$$\exists K > 0, \quad \forall k \geq 1, \quad \forall h \in \mathcal{V}, \quad (2\pi)^{-2} \int_{[-\pi, \pi]^2} \|P_u^k(h)\| du \leq \frac{K \|h\|}{k}. \quad (19)$$

The following result generalizes Proposition 3 of [16].

**Proposition 12.** *For any  $p > 1$ , there exist  $C > 0$  and  $b > 0$  such that for any nonnegative integers  $k, n, r, m$ , any  $N_1, N_2 \in \mathbb{Z}^2$ , any  $A_1, A_2, A_3 \subseteq \bar{M}$  union of components of  $\xi_{-k}^k$ , and any  $B \subseteq \bar{M}$  union of component of  $\xi_{-k}^\infty$ , we have*

$$\left| \text{Cov}_{\bar{\mu}}(\mathbf{1}_{A_1 \cap \{S_n = N_1\} \cap \bar{T}^{-n} A_2}, \mathbf{1}_{A_3 \cap \{S_r = N_2\} \cap \bar{T}^{-r} B} \circ \bar{T}^{n+m}) \right| \leq \frac{C \min(1, e^{-am+bk})}{nr}.$$

*Proof.* First, we assume that  $2k < \min(n, r)$  and  $m > 6k$ . Let us write

$$C_{n,m,r} := \text{Cov}_{\bar{\mu}}(\mathbf{1}_{A_1 \cap \{S_n = N_1\} \cap \bar{T}^{-n} A_2}, \mathbf{1}_{A_3 \cap \{S_r = N_2\} \cap \bar{T}^{-r} B} \circ \bar{T}^{n+m}).$$

Observe that  $\bar{T}^{-k} A_i$  is a union of components of  $\xi_0^{2k}$  and that  $\bar{T}^{-k} B$  is a union of components of  $\xi_0^\infty$ . Let  $\hat{A}_i := \hat{\pi} \tilde{\pi}^{-1} \bar{T}^{-k} A_i$  and  $\hat{B} := \hat{\pi} \tilde{\pi}^{-1} \bar{T}^{-k} B$ . These sets are measurable and satisfy  $\tilde{\pi}^{-1} \bar{T}^{-k} A_i = \hat{\pi}^{-1} \hat{A}_i$  and  $\tilde{\pi}^{-1} \bar{T}^{-k} B = \hat{\pi}^{-1} \hat{B}$ . So

$$\begin{aligned} C_{n,m,r} &= \text{Cov}_{\hat{\mu}}(\mathbf{1}_{\hat{A}_1} \mathbf{1}_{\hat{S}_n = N_1} \circ \hat{T}^k \mathbf{1}_{\hat{A}_2} \circ \hat{T}^n, (\mathbf{1}_{\hat{A}_3} \mathbf{1}_{\hat{S}_r = N_2} \circ \hat{T}^k \mathbf{1}_{\hat{B}} \circ \hat{T}^r) \circ \hat{T}^{n+m}) \\ &= \frac{1}{(2\pi)^4} \int_{([-\pi, \pi]^2)^2} e^{-i\langle u, N_1 \rangle} e^{-i\langle t, N_2 \rangle} \\ &\quad \times \text{Cov}_{\hat{\mu}}(\mathbf{1}_{\hat{A}_1} e^{i\langle u, \hat{S}_n \rangle} \circ \hat{T}^k \mathbf{1}_{\hat{A}_2} \circ \hat{T}^n, (\mathbf{1}_{\hat{A}_3} e^{i\langle t, \hat{S}_r \rangle} \circ \hat{T}^k \mathbf{1}_{\hat{B}} \circ \hat{T}^r) \circ \hat{T}^{n+m}) du dt. \end{aligned}$$

Now, due to (17), the covariance appearing in this last integral can be rewritten

$$\mathbb{E}_{\hat{\mu}}[P_t^k(\mathbf{1}_{\hat{B}} P_t^{r-k}(P^k(\mathbf{1}_{\hat{A}_3} P^{m-k}(g_u - \mathbb{E}_{\hat{\mu}}[g_u]))))],$$



with  $g_u := P_u^k(\mathbf{1}_{\hat{A}_2} P_u^{n-k}(P^k(\mathbf{1}_{\hat{A}_1})))$ . Since  $\|P_t\|_{L^1(\bar{\mu})} \leq 1$ , we obtain

$$\begin{aligned}
|C_{n,m,r}| &\leq (2\pi)^{-4} \int_{([- \pi, \pi]^2)^2} \mathbb{E}_{\bar{\mu}}[|\mathbf{1}_{\hat{B}} P_t^{r-k}(P^k(\mathbf{1}_{\hat{A}_3} P^{m-k}(g_u - \mathbb{E}_{\bar{\mu}}[g_u])))|] dt du \\
&\leq (2\pi)^{-4} \int_{([- \pi, \pi]^2)^2} C_0 \bar{\mu}(B)^{\frac{1}{p}} \|P_t^{r-k}(P^k(\mathbf{1}_{\hat{A}_3} P^{m-k}(g_u - \mathbb{E}_{\bar{\mu}}[g_u])))\| dt du \quad \text{by (13)} \\
&\leq (2\pi)^{-2} \int_{[- \pi, \pi]^2} \bar{\mu}(B)^{\frac{1}{p}} \frac{K C_0}{r-k} K_1(3\beta_0^{-2k}) \|P^{m-k}(g_u - \mathbb{E}_{\bar{\mu}}[g_u])\| du \quad \text{by (19)(18)(14)(15)} \\
&\leq (2\pi)^{-2} \bar{\mu}(B)^{\frac{1}{p}} \frac{K C_0}{r-k} K_1(3\beta_0^{-2k}) \int_{[- \pi, \pi]^2} K_0 e^{-a(m-k)} \|g_u\| du \quad \text{by (16)} \\
&\leq (2\pi)^{-2} \bar{\mu}(B)^{\frac{1}{p}} \frac{K C_0}{r-k} K_1(3\beta_0^{-2k}) K_0 e^{-a(m-k)} \int_{[- \pi, \pi]^2} \|P_u^k(\mathbf{1}_{\hat{A}_2} P_u^{n-k}(P^k(\mathbf{1}_{\hat{A}_1})))\| du \\
&\leq \bar{\mu}(B)^{\frac{1}{p}} \frac{K^2 C_0}{(r-k)(n-k)} (3K_1 \beta_0^{-2k})^3 K_0 e^{-a(m-k)} \quad \text{by (18)(19)(14)(15)} \\
&\leq \frac{\hat{C}_0 e^{-am+b_0 k}}{nr},
\end{aligned}$$

for some  $b_0 > 0$ . We still assume that  $m > 6k$ . When  $n \leq 2k$  and  $r > 2k$ , we observe that  $A_1 \cap \{S_n = N_1\} \cap \bar{T}^{-n} A_2$  is a union of components of  $\xi_{-k}^{3k}$ , using the same argument we obtain an upper bound in  $\hat{C}_0 e^{-am+3b_0 k}/r$  which is less than  $\hat{C}_1 e^{-am+4b_0 k}/(nr)$  for some  $\hat{C}_1 > 0$ . Treating analogously the cases  $(r \leq 2k; 2k < n)$  and  $(n \leq 2k; r \leq 2k)$ , we obtain the following bound

$$|C_{n,m,r}| \leq \frac{\hat{C} e^{-am+bk}}{nr}, \quad \text{for some } \hat{C} > 0 \text{ and some } b \geq 6a > 0. \quad (20)$$

Assume now that  $am \leq bk$  (this is true if  $m \leq 6k$ ). Then, due to the fact that  $|\text{Cov}_{\bar{\mu}}(f, g)| \leq |\mathbb{E}_{\bar{\mu}}[fg]| + |\mathbb{E}_{\bar{\mu}}[f]\mathbb{E}_{\bar{\mu}}[g]|$ , we have

$$\begin{aligned}
|C_{n,m,r}| &\leq \bar{\mu}(S_n = N_1; S_r \circ \bar{T}^{n+m} = N_2) + \bar{\mu}(S_n = N_1) \bar{\mu}(S_r = N_2) \\
&\leq |\text{Cov}_{\bar{\mu}}(\mathbf{1}_{S_n=N_1}, \mathbf{1}_{S_r=N_2} \circ \bar{T}^{n+m})| + 2\bar{\mu}(S_n = N_1) \bar{\mu}(S_r = N_2) \leq \frac{\hat{C}_2}{nr},
\end{aligned}$$

using estimation (20) with  $k = 0$  and the local limit theorem for  $S_n$  (see [20] or (6)).  $\square$

#### 4. ESTIMATE OF THE VARIANCE OF $V_n$

Recall that  $\Sigma^2$  is invertible. In particular, there exists  $\tilde{a}_0$  such that  $\langle (\Sigma^2)^{-1}x, x \rangle \geq 2\tilde{a}_0|x|^2$  for every  $x \in \mathbb{R}^2$ . Comparing

$$\sum_{x \in \mathbb{Z}^2 : |x| \leq am} e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2m}} \quad \text{with} \quad \int_{|u| \leq am} e^{-\frac{\langle (\Sigma^2)^{-1}u, u \rangle}{2m}} du,$$

we obtain the following useful formula

$$\sup_{\|S_1\|_{\infty} \leq a \leq 3\|S_1\|_{\infty}} \left| \sum_{x \in \mathbb{Z}^2 : |x| \leq am} e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2m}} - 2\pi m \sqrt{\det \Sigma^2} \right| = O(\sqrt{m}). \quad (21)$$

*Proof of Proposition 3.* As in [1], the proof of Proposition 3 is based on the following formula

$$\text{Var}(V_n) = 4 \sum_{1 \leq k_1 < \ell_1 \leq n} \sum_{1 \leq k_2 < \ell_2 \leq n} D_{k_1, \ell_1, k_2, \ell_2} = 8A_1 + 8A_2 + 8A_3 + 4A_4,$$

with  $D_{k_1, \ell_1, k_2, \ell_2} := \bar{\mu}(E_{k_1, \ell_1} \cap E_{k_2, \ell_2}) - \bar{\mu}(E_{k_1, \ell_1})\bar{\mu}(E_{k_2, \ell_2})$  and

$$\begin{aligned} A_1 &:= \sum_{1 \leq k_1 < k_2 < \ell_1 \leq \ell_2 \leq n} D_{k_1, \ell_1, k_2, \ell_2}, \quad A_2 := \sum_{1 \leq k_1 \leq k_2 < \ell_1 \leq \ell_2 \leq n} D_{k_1, \ell_1, k_2, \ell_2}, \\ A_3 &:= \sum_{1 \leq k_1 < k_2 < \ell_2 < \ell_1 \leq n} D_{k_1, \ell_1, k_2, \ell_2}, \quad A_4 := \sum_{1 \leq k_1 < \ell \leq n} [\bar{\mu}(E_{k_1, \ell}) - (\bar{\mu}(E_{k_1, \ell}))^2]. \end{aligned}$$

We use the notations and ideas of the proof of Proposition 10. Let  $p \in (1, 2)$ . We take  $m, k$  such that  $m^2 = \delta^{-k} = n^{1/100}$ . We have

$$\bar{\mu}(E_{k_1, \ell_1} \cap E_{k_2, \ell_2}) = \sum_{\mathcal{C}, \mathcal{C}' \in \mathcal{P}_m} \bar{\mu}(\mathcal{C} \cap E_{0, \ell_1 - k_1} \cap \bar{T}^{-(k_2 - k_1)}(\mathcal{C}' \cap E_{0, \ell_2 - k_2})).$$

As in the proof of Proposition 10, we approximate  $\mathcal{C} \cap E_{0, r}$  by  $\tilde{E}_{r, \mathcal{C}}$ . See (9) and (10) for the definition of  $\tilde{E}_{r, \mathcal{C}}$  and of  $\tilde{\mathcal{D}}_{r, \mathcal{C}}$ . We recall that  $(\mathcal{C} \cap E_{0, r}) \triangle \tilde{E}_{r, \mathcal{C}} \subseteq \tilde{\mathcal{D}}_{r, \mathcal{C}}$  and that, according to (7), if  $r \geq 3k$ , we have (for  $p > 1$  large enough)

$$\bar{\mu}(\tilde{E}_{r, \mathcal{C}}) = O\left(\frac{m^{-2}}{r} + \frac{km^{-2/p}}{r^{\frac{3}{2}}}\right) = O(m^{-2}r^{-1}) = O(r^{-1}n^{-\frac{1}{100}}) \quad (22)$$

and

$$\bar{\mu}(\tilde{\mathcal{D}}_{r, \mathcal{C}}) \leq \frac{m^{-2}\delta^k}{r} + \frac{km^{-2/p}}{r^{\frac{3}{2}}} = O(m^{-2}r^{-1}\delta^k) = O(r^{-1}n^{-\frac{2}{100}}). \quad (23)$$

- Control of  $A_1$ . We have

$$\begin{aligned} & |\text{Cov}_{\bar{\mu}}(\mathbf{1}_{\mathcal{C} \cap E_{0, r}}, \mathbf{1}_{\mathcal{C}' \cap E_{0, s}} \circ \bar{T}^{r+\ell}) - \text{Cov}_{\bar{\mu}}(\mathbf{1}_{\tilde{E}_{r, \mathcal{C}}}, \mathbf{1}_{\tilde{E}_{s, \mathcal{C}'}} \circ \bar{T}^{r+\ell})| \leq \\ & \leq |\text{Cov}_{\bar{\mu}}(\mathbf{1}_{\tilde{E}_{r, \mathcal{C}} \cup \tilde{\mathcal{D}}_{r, \mathcal{C}}}, \mathbf{1}_{\tilde{\mathcal{D}}_{s, \mathcal{C}'}} \circ \bar{T}^{r+\ell})| + |\text{Cov}_{\bar{\mu}}(\mathbf{1}_{\tilde{\mathcal{D}}_{r, \mathcal{C}}}, \mathbf{1}_{\tilde{E}_{s, \mathcal{C}'}} \circ \bar{T}^{r+\ell})| \\ & \quad + 2\bar{\mu}(\tilde{E}_{r, \mathcal{C}} \cup \tilde{\mathcal{D}}_{r, \mathcal{C}})\bar{\mu}(\tilde{\mathcal{D}}_{s, \mathcal{C}'}) + 2\bar{\mu}(\tilde{\mathcal{D}}_{r, \mathcal{C}})\bar{\mu}(\tilde{E}_{s, \mathcal{C}'}) + 2\bar{\mu}(\tilde{\mathcal{D}}_{r, \mathcal{C}})\bar{\mu}(\tilde{E}_{s, \mathcal{C}'}) + 2\bar{\mu}(\tilde{\mathcal{D}}_{r, \mathcal{C}})\bar{\mu}(\tilde{\mathcal{D}}_{s, \mathcal{C}'})). \end{aligned}$$

Now, due to (11), (12), applying Proposition 12 (together with (22) and (23)), we obtain

$$\sum_{\mathcal{C}} \sum_{\mathcal{C}'} |\text{Cov}_{\bar{\mu}}(\mathbf{1}_{\mathcal{C} \cap E_{0, r}}, \mathbf{1}_{\mathcal{C}' \cap E_{0, s}} \circ \bar{T}^{r+\ell})| \leq m^4 \frac{C \min(1, e^{-al+bk})}{rs} + \frac{Cn^{-\frac{1}{100}}}{rs},$$

and so (considering separately the sums over  $\ell$  such that  $al \geq 2bk$  and  $al < 2bk$ )

$$A_1 = \sum_{k_1 \geq 1, r > 0, \ell \geq 1, s > 0: k_1 + r + \ell + s \leq n} \text{Cov}_{\bar{\mu}}(\mathbf{1}_{E_{0, r}}, \mathbf{1}_{E_{0, s}} \circ \bar{T}^{r+\ell}) = O(n^{2-\frac{1}{100}} \log^2 n). \quad (24)$$

- Control of  $A_2$ . Notice that

$$A_2 = \sum_{k_1 + r + \ell + s \leq n} \text{Cov}_{\bar{\mu}}(E_{0, r+\ell}, E_{r, r+\ell+s})$$

(where the sum is also taken over  $k_1 \geq 1, r \geq 0, \ell \geq 1, s \geq 0$ ). According to Proposition 10, we have  $\bar{\mu}(E_{0, r}) = \frac{c}{2r} + O(r^{-1-\eta})$  with  $\eta > 0$ . A direct computation (see Lemma 13) gives  $\sum_{k_1 + r + \ell + s \leq n} \frac{1}{(r+\ell)(\ell+s)} \sim \frac{\pi^2}{12} n^2$ . Hence

$$\sum_{k_1 + r + \ell + s \leq n} \bar{\mu}(E_{0, r+\ell})\bar{\mu}(E_{0, \ell+s}) \sim \frac{\pi^2}{12} \frac{c^2}{4} n^2. \quad (25)$$

Now, let us prove that

$$\sum_{k_1 + r + \ell + s \leq n} \bar{\mu}(E_{0, r+\ell} \cap E_{r, r+\ell+s}) \sim J \frac{c^2}{4} n^2. \quad (26)$$

From which we conclude that

$$A_2 \sim \left(J - \frac{\pi^2}{12}\right) \frac{c^2}{4} n^2. \quad (27)$$

We have to estimate  $C_{r,\ell,s}^{(2)} := \bar{\mu}(E_{0,r+\ell} \cap E_{r,r+\ell+s})$ . Given  $\mathcal{C}, \mathcal{C}' \in \mathcal{P}_m$ , we consider the set  $\mathcal{E}_{r,\ell,s,\mathcal{C},\mathcal{C}'}^{(2)} := E_{0,r+\ell} \cap E_{r,r+\ell+s} \cap \mathcal{C} \cap \bar{T}^{-r}\mathcal{C}'$  which we approximate by  $\tilde{\mathcal{E}}_{r,\ell,s,\mathcal{C},\mathcal{C}'}^{(2)} := \tilde{E}_{r+\ell,\mathcal{C}} \cap \bar{T}^{-r}\tilde{E}_{\ell+s,\mathcal{C}'}$ . We notice that

$$\mathcal{E}_{r,\ell,s,\mathcal{C},\mathcal{C}'}^{(2)} \Delta \tilde{\mathcal{E}}_{r,\ell,s,\mathcal{C},\mathcal{C}'}^{(2)} \subseteq (\tilde{\mathcal{D}}_{r+\ell,\mathcal{C}} \cap \bar{T}^{-r}(\tilde{\mathcal{D}}_{\ell+s,\mathcal{C}'} \cup \tilde{E}_{\ell+s,\mathcal{C}'})) \cup ((\tilde{E}_{r+\ell,\mathcal{C}} \cup \tilde{\mathcal{D}}_{r+\ell,\mathcal{C}}) \cap \bar{T}^{-r}\tilde{\mathcal{D}}_{\ell+s,\mathcal{C}'}). \quad (28)$$

Observe that  $\bar{\mu}(\tilde{\mathcal{E}}_{r,\ell,s,\mathcal{C},\mathcal{C}'}^{(2)})$  is equal to the following sum  $\sum_x G_{r,\ell,s,\mathcal{C},\mathcal{C}',x}$  (where  $\sum_x$  means the sum over the  $x \in \mathbb{Z}^2$  such that  $|x| \leq \min(r, \ell + 1, s + 2)\|S_1\|_\infty$ ) with

$$G_{r,\ell,s,\mathcal{C},\mathcal{C}',x} := \sum_{|N|,|N'| \leq \|S_1\|_\infty} \bar{\mu}(\tilde{\mathcal{C}} \cap \{S_r = x\} \cap \bar{T}^{-r}(\tilde{\mathcal{C}}' \cap \{S_\ell = N - x\} \cap \bar{T}^{-\ell}(\bar{M} \cap (\tilde{V} - N) \cap \{S_s = x + N' - N\} \cap \bar{T}^{-s}(\bar{M} \cap (\tilde{V}' - N'))))), \quad (29)$$

and where  $\tilde{\mathcal{C}}, \tilde{\mathcal{C}}', \tilde{V}$  and  $\tilde{V}'$  are the  $\xi_{-k}^k$ -measurable sets such that  $\tilde{E}_{r+\ell,\mathcal{C}} = \tilde{\mathcal{C}} \cap T^{-r-\ell}\tilde{V}$  and  $\tilde{E}_{\ell+s,\mathcal{C}'} = \tilde{\mathcal{C}}' \cap T^{-\ell-s}\tilde{V}'$  (see (9)). Due to (28), we have

$$\bar{\mu}(\mathcal{E}_{r,\ell,s,\mathcal{C},\mathcal{C}'}^{(2)} \Delta \tilde{\mathcal{E}}_{r,\ell,s,\mathcal{C},\mathcal{C}'}^{(2)}) \leq \sum_x (\mathcal{S}_{1,x} + \mathcal{S}_{2,x}), \quad (30)$$

where  $\mathcal{S}_{1,x}$  (resp.  $\mathcal{S}_{2,x}$ ) is obtained from (29) by replacing  $\tilde{V}$  and  $\tilde{V}'$  by  $\tilde{\mathcal{D}}_{\mathcal{C}}$  and  $\tilde{V}' \cup \tilde{\mathcal{D}}_{\mathcal{C}'}$  (resp. by  $\tilde{V} \cup \tilde{\mathcal{D}}_{\mathcal{C}}$  and  $\tilde{\mathcal{D}}_{\mathcal{C}'}$ ), with the notation  $\tilde{\mathcal{D}}_{\mathcal{C}}$  introduced in (10). To estimate  $\bar{\mu}(\tilde{\mathcal{E}}_{r,\ell,s,\mathcal{C},\mathcal{C}'}^{(2)})$  and  $\bar{\mu}(\mathcal{E}_{r,\ell,s,\mathcal{C},\mathcal{C}'}^{(2)} \Delta \tilde{\mathcal{E}}_{r,\ell,s,\mathcal{C},\mathcal{C}'}^{(2)})$ , we will apply (6) three successive times to each summand appearing in (29) and in (30).

We start with the study of (29). According to (6) and since  $|N|, |N'| \leq \|S_1\|_\infty$ , when  $r, \ell, s \geq 3k$ , the quantity given by (29) is equal to

$$\frac{\bar{\mu}(\tilde{\mathcal{C}})\bar{\mu}(\odot)}{\sqrt{\det \Sigma^2 2\pi r}} e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2r}} + e_1, \quad (31)$$

with

$$\bar{\mu}(\odot) = \frac{\bar{\mu}(\tilde{\mathcal{C}}')\bar{\mu}(\otimes)}{\sqrt{\det \Sigma^2 2\pi \ell}} e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2\ell}} + e_2, \quad (32)$$

$$\bar{\mu}(\otimes) = \frac{\bar{\mu}(\tilde{V})\bar{\mu}(\tilde{V}')}{\sqrt{\det \Sigma^2 2\pi s}} e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2s}} + e_3, \quad (33)$$

the error terms being estimated by

$$|e_1| \leq \tilde{K}_1 k \frac{\bar{\mu}(\odot)^{\frac{1}{p}}}{r^{\frac{3}{2}}}, \quad |e_2| \leq \tilde{K}_1 k \frac{\bar{\mu}(\otimes)^{\frac{1}{p}}}{\ell^{\frac{3}{2}}} \quad \text{and} \quad |e_3| \leq \tilde{K}_1 k s^{-\frac{3}{2}},$$

for some  $\tilde{K}_1 > 1$ . So the contribution to  $A_2$  of the three dominating terms in (31), (32) and (33) is (where  $\sum^+$  means the sum restricted to  $k_1 \geq 1, \min(r, s, \ell) \geq 3k$ ):

$$\begin{aligned} & \sum_{k_1+r+\ell+s \leq n}^+ \sum_x \sum_{\mathcal{C}, \mathcal{C}'} \frac{\bar{\mu}(\tilde{\mathcal{C}})\bar{\mu}(\tilde{\mathcal{C}}')\bar{\mu}(\tilde{V})\bar{\mu}(\tilde{V}')}{(\det \Sigma^2)^{\frac{3}{2}}(2\pi)^3 r \ell s} e^{-\frac{1}{2}\langle (\Sigma^2)^{-1}x, x \rangle (\frac{1}{r} + \frac{1}{\ell} + \frac{1}{s})} \\ &= \sum_{\mathcal{C}, \mathcal{C}'} \frac{4\mathbb{E}_{\bar{\mu}}[\tau \mathbf{1}_{\mathcal{C}}]\mathbb{E}_{\bar{\mu}}[\tau \mathbf{1}_{\mathcal{C}'}](1 + o(1))}{(\det \Sigma^2)^{\frac{3}{2}}(2\pi)^3 (\sum_i |\partial O_i|)^2} \sum_{k_1+r+\ell+s \leq n}^+ \sum_x \frac{e^{-\frac{1}{2}\langle (\Sigma^2)^{-1}x, x \rangle (\frac{1}{r} + \frac{1}{\ell} + \frac{1}{s})}}{r \ell s}. \end{aligned}$$

Since  $1/\min(r, \ell, s) \leq \frac{1}{r} + \frac{1}{\ell} + \frac{1}{s} \leq 3/\min(r, \ell, s)$ , due to (21), we have

$$\begin{aligned} \sum_{k_1+r+\ell+s \leq n}^+ \sum_x \frac{e^{-\frac{1}{2} \langle (\Sigma^2)^{-1} x, x \rangle (\frac{1}{r} + \frac{1}{\ell} + \frac{1}{s})}}{2\pi \sqrt{\det \Sigma^2} r \ell s} &= \sum_{k_1+r+\ell+s \leq n}^+ \left( \frac{1}{r\ell + rs + s\ell} + \frac{O\left(\sqrt{\min(r, \ell, s)}\right)}{r\ell s} \right) \\ &= O(n^{\frac{3}{2}}) + \sum_{k_1+r+\ell+s \leq n}^+ \frac{1}{r\ell + rs + s\ell} \\ &= o(n^2) + \sum_{k_1+r+\ell+s \leq n} \frac{1}{r\ell + rs + s\ell}, \end{aligned}$$

(where the last sum is taken over  $k_1, r, \ell, s \geq 1$ ) since

$$\sum_{k_1, r, s=1}^n \sum_{\ell=1}^{3k} \frac{1}{r\ell + rs + s\ell} \leq O(n \log n) \sum_{r, s=1}^n \frac{1}{rs} = O(n \log^3 n) = o(n^2).$$

Now, according to Lemma 14, we have

$$\sum_{k_1+r+\ell+s \leq n} \frac{1}{r\ell + rs + s\ell} \sim n^2 J.$$

We finally obtain that the contribution to  $A_2$  of (29) coming from the dominating terms of (31), (32) and (33) is

$$\sim J \frac{c^2}{4} n^2. \quad (34)$$

Now, we prove that the other contributions are in  $o(n^2)$ .

- Using the fact that  $|x| \leq 2 \min(r, \ell, s) \|S_1\|_\infty$ , we get that the contribution to  $A_2$  of the term coming from the composition of the three error terms  $(e_1, e_2, e_3)$  is bounded by

$$\begin{aligned} m^4 \sum_{k_1+r+\ell+s \leq n} \sum_x \frac{\tilde{K}_1^3 k^3}{r^{\frac{3}{2}} \ell^{\frac{3}{2p}} s^{\frac{3}{2p^2}}} &\leq 16(\|S_1\|_\infty + 1)^4 \tilde{K}_1^3 n^{\frac{2}{100}} n k^3 \sum_{r+\ell+s \leq n} \frac{\min(r^2, \ell^2, s^2)}{r^{\frac{3}{2}} \ell^{\frac{3}{2p}} s^{\frac{3}{2p^2}}}. \\ &\leq O\left(n^{\frac{102}{100}} \log^3 n \sum_{r, \ell, s \leq n} \frac{r^{\frac{2}{3}} \ell^{\frac{2}{3}} s^{\frac{2}{3}}}{r^{\frac{3}{2}} \ell^{\frac{3}{2p}} s^{\frac{3}{2p^2}}}\right) = O(n^{\frac{102}{100} + 5 - \frac{9}{2p^2}}) = o(n^2), \end{aligned}$$

if we take  $p > 1$  small enough.

- Analogously, the contribution to  $A_2$  of the composition of one dominating term and of two error terms of (31), (32) and (33) is less (up to a multiplicative constant) than

$$\begin{aligned} n^{\frac{2}{100}} k^2 \sum_{k_1+r+\ell+s \leq n} \frac{1}{r^{\frac{1}{p^2}} \ell^{\frac{3}{2}} s^{\frac{3}{2p}}} \sum_x e^{-\tilde{a}_0^2 \frac{|x|^2}{rp^2}} &\leq O\left(n^{\frac{102}{100}} k^2 \sum_{r, \ell, s \leq n} \frac{\min(r, \ell^2, s^2)}{r^{\frac{1}{p^2}} \ell^{\frac{3}{2}} s^{\frac{3}{2p}}}\right) \\ &\leq O\left(n^{\frac{102}{100}} k^2 \sum_{r, \ell, s \leq n} \frac{r^{\frac{1}{2}} (\ell^2)^{\frac{1}{4}} (s^2)^{\frac{1}{4}}}{r^{\frac{1}{p^2}} \ell^{\frac{3}{2}} s^{\frac{3}{2p}}}\right) \\ &= O(n^{\frac{102}{100} + \frac{3}{2} - \frac{1}{p^2} + \frac{3}{2} - \frac{3}{2p}} \log^3 n) = o(n^2), \end{aligned}$$

if we take  $p > 1$  small enough.

- Now, the contribution to  $A_2$  of the composition of two dominating terms and of one error term of (31), (33) and (33) is less (up to a multiplicative constant) than

$$n^{\frac{102}{100}} \sum_{r, \ell, s \leq n} \frac{k^2}{r^{\frac{1}{p}} \ell^{\frac{1}{p}} s^{\frac{3}{2}}} \sum_x e^{-\frac{\tilde{a}_0^2}{p} |x|^2 \frac{r+\ell}{r\ell}}.$$

On the one hand, we have  $\sum_x e^{-a_0^2 |x|^2 \frac{r+\ell}{r\ell}} \leq \min(r, \ell)$  (using the fact that  $r\ell/(r+\ell) \leq \min(r, \ell)$ ). On the other hand, this sum is in  $O(s^2)$ . Therefore the quantity we are looking at is less than

$$\begin{aligned} O \left( n^{\frac{102}{100}} k^2 \sum_{r, \ell, s \leq n} \frac{\min(r, \ell, s^2)}{r^{\frac{1}{p}} \ell^{\frac{1}{p}} s^{\frac{3}{2}}} \right) &= O \left( n^{\frac{102}{100}} k^2 \sum_{r, \ell, s \leq n} \frac{r^{\frac{3}{8}} \ell^{\frac{3}{8}} (s^2)^{\frac{1}{4}}}{r^{\frac{1}{p}} \ell^{\frac{1}{p}} s^{\frac{3}{2}}} \right) \\ &= O \left( n^{\frac{102}{100} + 2 + \frac{3}{4} - \frac{2}{p}} \log^3 n \right) = o(n^2), \end{aligned}$$

if  $p > 1$  is small enough.

- If  $r \leq 4k$  or  $\ell \leq 4k$  or  $s \leq 4k$ , then  $\sum_x$  is a sum over  $|x| \leq 4k \|S_1\|_\infty$  and one of the following sets is  $\xi_{-5k}^{5k}$ -measurable:

$$\begin{aligned} &\tilde{\mathcal{C}} \cap \{S_r = x\} \cap \bar{T}^{-r} \tilde{\mathcal{C}}' \quad \text{or} \quad \tilde{\mathcal{C}}' \cap \{S_\ell = N - x\} \cap \bar{T}^{-\ell} (\bar{M} \cap (\tilde{V} - N)) \\ &\quad \text{or} \quad \bar{M} \cap (\tilde{V} - N) \cap \{S_s = x + N' - N\} \cap \bar{T}^{-s} (\bar{M} \cap (\tilde{V}' - N')). \end{aligned}$$

We then apply (6) accordingly and take in account the fact that the sum on  $r$  or  $k$  or  $\ell$  must be taken on  $\{1, \dots, 4k\}$ . This leads to a term in  $o(n^2)$ .

- Finally, the estimate of (30) follows the same lines as the estimate of (29). We obtain an analogous estimation multiplied by  $\delta^k$ . This ensures that the contribution of (30) to  $A_2$  is in  $o(n^2)$ .

- Control of  $A_3$ . We have

$$A_3 = \sum_{k_1+r+\ell+s \leq n} \text{Cov}_{\bar{\mu}}(\mathbf{1}_{E_{0,r+\ell+s}}, \mathbf{1}_{E_{0,\ell}} \circ \bar{T}^r).$$

This part is the most delicate. Indeed the terms

$$\sum_{k_1+r+\ell+s \leq n} \bar{\mu}(E_{0,r+\ell+s}) \bar{\mu}(E_{0,\ell}) \quad \text{and} \quad \sum_{k_1+r+\ell+s \leq n} \bar{\mu}(E_{0,r+\ell+s} \cap E_{r,r+\ell})$$

are in  $n^2 \log n$ . But we will prove that their difference is in  $n^2$ . More precisely, we show that

$$A_3 \sim \frac{c^2}{8} n^2. \tag{35}$$

First, according to Proposition 10, we have

$$\begin{aligned} \sum_{k_1+r+\ell+s \leq n} \bar{\mu}(E_{0,r+\ell+s}) \bar{\mu}(E_{0,\ell}) &= \sum_{k_1+r+\ell+s \leq n} \left( \frac{c + O((r+\ell+s)^{-\eta})}{2(r+\ell+s)} \right) \bar{\mu}(E_{0,\ell}) \\ &= o(n^2) + \sum_{\mathcal{C}} \sum_{k_1+r+\ell+s \leq n} \frac{c \bar{\mu}(E_{0,\ell} \cap \mathcal{C})}{2(r+\ell+s)} \\ &= o(n^2) + \sum_{\mathcal{C}} \sum_{k_1+r+\ell+s \leq n} \frac{c \bar{\mu}(\tilde{E}_{\ell, \mathcal{C}}) + O(n^{-\frac{2}{100}}/\ell)}{2(r+\ell+s)} \\ &= o(n^2) + \sum_{\mathcal{C}} \sum_{k_1+r+\ell+s \leq n} \frac{c \bar{\mu}(\tilde{E}_{\ell, \mathcal{C}})}{2(r+\ell+s)}. \end{aligned} \tag{36}$$

Indeed, setting  $q = \ell + r$  and  $t = \ell + r + s$ , we have

$$\sum_{\mathcal{C}} \sum_{k_1+r+\ell+s \leq n} \frac{n^{-\frac{2}{100}}}{\ell(r+\ell+s)} \leq n^{1-\frac{1}{100}} \sum_{t=1}^n \frac{1}{t} \sum_{q=1}^t \sum_{\ell=1}^q \frac{1}{\ell} = O(n^{2-\frac{1}{100}} \log n) = o(n^2).$$

Now, let us estimate  $\sum_{k_1+r+\ell+s \leq n} \bar{\mu}(E_{0,r+\ell+s} \cap E_{r,r+\ell})$  in terms of  $\bar{\mu}(\tilde{E}_{\ell,\mathcal{C}})$ . For any  $\mathcal{C}, \mathcal{C}' \in \mathcal{P}_m$ , we approximate once again  $\mathcal{C}' \cap E_{0,r+\ell+s} \cap \bar{T}^{-r}\mathcal{C} \cap E_{r,r+\ell}$  by

$$\tilde{E}_{r+\ell+s,\mathcal{C}'} \cap \bar{T}^{-r} \tilde{E}_{\ell,\mathcal{C}}, \quad (37)$$

the measure of which is  $\sum_x H_{r,\ell,s,\mathcal{C},\mathcal{C}',x}$  (with  $\sum_x$  being taken on the set of  $x \in \mathbb{Z}^2$  such that  $|x| \leq \min(r, s+2)\|S_1\|_\infty$ ) and with

$$H_{r,\ell,s,\mathcal{C},\mathcal{C}',x} := \sum_{|N|,|N'| \leq L} \bar{\mu}(\tilde{\mathcal{C}}' \cap \{S_r = x\} \cap \bar{T}^{-r}(\tilde{\mathcal{C}} \cap \{S_\ell = N\} \cap \bar{T}^{-\ell}(\bar{M} \cap (\tilde{V} - N) \cap \{S_s = N' - x - N\} \cap \bar{T}^{-s}(\bar{M} \cap (\tilde{V}' - N'))))). \quad (38)$$

Now, applying (6) and (7) (when  $\min(r, s) \geq 3k$ ), we obtain that this quantity is equal to

$$\frac{\bar{\mu}(\tilde{\mathcal{C}}')\bar{\mu}(\diamond)}{\sqrt{\det \Sigma^2} 2\pi r} e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2r}} + e'_1, \quad (39)$$

with

$$\bar{\mu}(\diamond) = \frac{\bar{\mu}(\tilde{E}_{\ell,\mathcal{C}})\bar{\mu}(\tilde{V}')}{\sqrt{\det \Sigma^2} 2\pi s} e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2s}} + e'_2, \quad (40)$$

the error terms being estimated by

$$|e'_1| \leq \tilde{K}_1 k \frac{\bar{\mu}(\diamond)^{\frac{1}{p}}}{r^{\frac{3}{2}}} \quad \text{and} \quad |e'_2| \leq \tilde{K}_1 k \frac{\bar{\mu}(\tilde{E}_{\ell,\mathcal{C}})^{\frac{1}{p}}}{s^{\frac{3}{2}}}.$$

We obtain that the contribution to  $A_3$  of the dominating terms of (36), (39) and (40) is (where  $\sum^*$  stands for the sum over  $k_1 \geq 1$ ,  $\ell \geq 1$  and  $\min(r, s) \geq 3k$ )

$$\begin{aligned} & \sum_{\mathcal{C}, \mathcal{C}'} \sum_{k_1+r+\ell+s \leq n}^* \left( \frac{\bar{\mu}(\tilde{\mathcal{C}}')\bar{\mu}(\tilde{V}')\bar{\mu}(\tilde{E}_{\ell,\mathcal{C}}) \sum_x e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2} \frac{r+s}{rs}}}{\det \Sigma^2 (2\pi)^2 rs} - \frac{c\bar{\mu}(\tilde{\mathcal{C}}')\bar{\mu}(\tilde{E}_{\ell,\mathcal{C}})}{2(r+\ell+s)} \right) \\ &= \sum_{k_1+r+\ell+s \leq n}^* \sum_{\mathcal{C}} \bar{\mu}(\tilde{E}_{\ell,\mathcal{C}}) \frac{c}{2} \left( \frac{(1 + O(n^{-\frac{1}{200}})) \sum_x e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2} \frac{r+s}{rs}}}{\sqrt{\det \Sigma^2} 2\pi rs} - \frac{1}{r+\ell+s} \right) \\ &= o(n^2) + \sum_{k_1+r+\ell+s \leq n}^* \frac{c^2}{4\ell} \left( \frac{\sum_x e^{-\frac{\langle (\Sigma^2)^{-1}x, x \rangle}{2} \frac{r+s}{rs}}}{\sqrt{\det \Sigma^2} 2\pi rs} - \frac{1}{r+\ell+s} \right) \\ &= o(n^2) + \sum_{k_1+r+\ell+s \leq n}^* \frac{c^2}{4\ell} \left( \frac{1}{r+s} - \frac{1}{r+\ell+s} \right) \quad \text{due to (21)} \\ &= o(n^2) + \frac{c^2}{4} \sum_{k_1+r+\ell+s \leq n}^* \left( \frac{1}{(r+s)(r+\ell+s)} \right) \\ &= o(n^2) + \frac{c^2}{4} \sum_{k_1, r, \ell, s \geq 1 : k_1+r+\ell+s \leq n} \left( \frac{1}{(r+s)(r+\ell+s)} \right) \\ &\sim \frac{c^2}{4} n^2 \int_{[0,1]^4} \frac{\mathbf{1}_{\{t+u+v+w \leq 1\}} dt du dv dw}{(u+w)(u+v+w)} = \frac{c^2}{8} n^2. \end{aligned}$$

For the third line, we used the fact that  $\sum_{\mathcal{C}} \bar{\mu}(\tilde{E}_{\ell, \mathcal{C}}) = \frac{c}{2\ell} + O(\ell^{-1-\eta})$ . For the last line, we used the Lebesgue dominated convergence theorem and the following equalities obtained by a change of variable ( $r = u + w$ ,  $s = u + v + w$ ) and by integrating in  $t$ ,  $u$ ,  $r$  and finally in  $s$ :

$$\begin{aligned} \int_{[0,1]^4} \frac{\mathbf{1}_{\{t+u+v+w < 1\}} dt du dv dw}{(u+w)(u+v+w)} &= \int_{[0,1]^4} \frac{\mathbf{1}_{\{u < r < s, t+s < 1\}} dt du dr ds}{rs} \\ &= \int_{0 \leq u \leq r \leq s \leq 1} \frac{(1-s) du dr ds}{rs} \\ &= \int_0^1 (1-s) ds = \frac{1}{2}. \end{aligned}$$

Now, it remains to show that the contribution to  $A_3$  of all the other terms is in  $o(n^2)$ .

- According to (22), (39) and (40), the contribution of the term coming from the composition of the two error terms  $e'_1$  and  $e'_2$  is in

$$\begin{aligned} \sum_{\mathcal{C}, \mathcal{C}'} \sum_{k_1+r+\ell+s \leq n} \sum_x k^2 \frac{\bar{\mu}(\tilde{E}_{\ell, \mathcal{C}})^{\frac{1}{p^2}}}{r^{\frac{3}{2}} s^{\frac{3}{2p}}} &= 4 \sum_{\mathcal{C}, \mathcal{C}'} \sum_{k_1+r+\ell+s \leq n} \frac{k^2 n^{-\frac{1}{100p^2}}}{\ell^{\frac{1}{p^2}} r^{\frac{3}{2}} s^{\frac{3}{2p}}} \min(r^2, s^2) \quad (41) \\ &= 4 \sum_{\mathcal{C}, \mathcal{C}'} \sum_{k_1+r+\ell+s \leq n} \frac{k^2 n^{-\frac{1}{100p^2}}}{\ell^{\frac{1}{p^2}} r^{\frac{3}{2}} s^{\frac{3}{2p}}} rs \\ &= O(n^{1+\frac{1}{100}(2-\frac{1}{p^2})+\frac{1}{2}+1-\frac{1}{p^2}+2-\frac{3}{2p}} \log^2 n), \end{aligned}$$

which is not enough to conclude. Hence, we use the estimate of  $e'_2$  given by Remark 9 for  $x \geq 3k$ . On the one hand, the last term in the RHS of the formula given in Remark 9 brings (41) with  $s^{\frac{3}{2p}}$  replaced by  $ks^{\frac{2}{p}}$ , which gives  $o(n^2)$  for  $p > 1$  small enough. On the other hand, the first term in the RHS of the formula of Proposition 8 gives still  $s^{\frac{3}{2p}}$ , but with  $\min(r^2, s) \leq s^{\frac{1}{2}}r$  instead of  $\min(r^2, s^2) \leq rs$ . This ensures that this term is in  $o(n^2)$ .

- The contribution of the term coming from the composition of the error term  $e'_1$  of (39) and of the dominating term of (40) is in

$$\begin{aligned} \sum_{\mathcal{C}, \mathcal{C}'} \sum_{k_1+r+\ell+s \leq n} \sum_x \frac{k}{r^{\frac{3}{2}}} \left( \frac{\bar{\mu}(\tilde{E}_{\ell, \mathcal{C}}) \bar{\mu}(\tilde{V}') e^{-\tilde{a}_0 \frac{|x|^2}{s}}}{s} \right)^{\frac{1}{p}} \\ = O \left( n^{\frac{2}{100}} \log n \sum_{k_1+r+\ell+s \leq n} \frac{\min(s, r^2)}{r^{\frac{3}{2}} \ell^{\frac{1}{p}} s^{\frac{1}{p}}} \right) \\ = O \left( n^{1+\frac{2}{100}} \log n \sum_{r+\ell+s \leq n} \frac{\sqrt{r} s^{\frac{3}{4}}}{r^{\frac{3}{2}} \ell^{\frac{1}{p}} s^{\frac{1}{p}}} \right) \\ = O \left( (\log n)^2 n^{1+\frac{2}{100}+1-\frac{1}{p}+\frac{7}{4}-\frac{1}{p}} \right) = o(n^2), \end{aligned}$$

if  $p > 1$  is small enough (using the fact that  $\sum_x e^{-\tilde{a}_0 \frac{|x|^2}{ps}} = O(\min(s, r^2))$ ).

- Now, the contribution of the term coming from the composition of the dominating term of (39) and of the error term  $e'_2$  term of (40) is in

$$\begin{aligned}
\sum_{\mathcal{C}, \mathcal{C}'} \sum_{k_1+r+\ell+s \leq n} \sum_x \frac{\bar{\mu}(\tilde{\mathcal{C}}')}{r} e^{-\frac{\langle (\Sigma^2)^{-1} x, x \rangle}{2r}} k \frac{\bar{\mu}(\tilde{E}_{\ell, \mathcal{C}})^{\frac{1}{p}}}{s^{\frac{3}{2}}} &= \\
&= n^{\frac{1}{100}} \log n \sum_{k_1+r+\ell+s \leq n} \frac{\min(r, s^2)}{r \ell^{\frac{1}{p}} s^{\frac{3}{2}}} \\
&= n^{\frac{1}{100}} \log n \sum_{k_1+r+\ell+s \leq n} \frac{r^{\frac{3}{4}} (s^2)^{\frac{1}{4}}}{r \ell^{\frac{1}{p}} s^{\frac{3}{2}}} \\
&= n^{1+\frac{1}{100}+\frac{3}{4}+1-\frac{1}{p}} \log^2 n = o(n^2),
\end{aligned}$$

if  $p > 1$  is small enough.

- For the control of the sum over  $(k_1, r, s, \ell)$  such that  $\min(r, s) < 3k$ , we proceed as we did for  $A_2$ .
- It remains to estimate

$$\sum_{\mathcal{C}, \mathcal{C}'} \sum_{k_1+r+\ell+s \leq n} (\bar{\mu}(\tilde{\mathcal{D}}_{r+\ell+s, \mathcal{C}'} \cap T^{-r}(\tilde{E}_{\ell, \mathcal{C}} \cup \tilde{\mathcal{D}}_{\ell, \mathcal{C}})) + \bar{\mu}((\tilde{E}_{r+\ell+s, \mathcal{C}'} \cup \tilde{\mathcal{D}}_{r+\ell+s, \mathcal{C}'} \cap T^{-r} \tilde{\mathcal{D}}_{\ell, \mathcal{C}})).$$

The dominating terms obtained by (6) are estimated as the dominating terms of (39) and (40). They bring a contribution to  $A_3$  in

$$\delta^k \sum_{k_1+r+\ell+s \leq n} \frac{1}{(r+s)\ell} \leq \delta^k n^2 \log n = o(n^2).$$

The fact that the other terms are in  $o(n^2)$  follows as for the study of (37).

- Control of  $A_4$ . We have  $A_4 \leq \sum_{1 \leq k_1 < \ell \leq n} \mathbb{P}(E_{k_1, \ell}) = O(n \log n) = o(n^2)$ .

Finally we have  $\text{Var}_{\bar{\mu}}(V_n) \sim 8(A_2 + A_3)$ . □

**Lemma 13.** *We have*

$$\sum_{k_1 \geq 1, r \geq 0, \ell \geq 1, s \geq 0: k_1+r+\ell+s \leq n} \frac{1}{(r+\ell)(\ell+s)} \sim \frac{\pi^2}{12} n^2.$$

*Proof.* Comparing the sum with an integral (by the Lebesgue dominated convergence theorem) and making the change of variables  $r = \min(u+v, u+w)$  and  $s = \max(u+v, u+w)$ , we obtain

$$\begin{aligned}
\sum_{r+\ell+s \leq n} \frac{1}{(r+\ell)(\ell+s)} &\sim n \int_{\{u, v, w > 0 : u+v+w \leq 1\}} \frac{du dv dw}{(u+v)(u+w)} \\
&\sim 2n \int_0^1 \left( \int_u^{\frac{1+u}{2}} \frac{1}{r} \left( \int_r^{1-r+u} \frac{ds}{s} \right) dr \right) du \\
&\sim 2n \int_0^1 \left( \int_u^{\frac{1+u}{2}} \frac{1}{r} \log \left( \frac{1+u}{r} - 1 \right) dr \right) du.
\end{aligned}$$

But

$$\int_u^{\frac{1+u}{2}} \frac{1}{r} \log \left( \frac{1+u}{r} - 1 \right) dr = \int_2^{1+\frac{1}{u}} \frac{\log(w-1)}{w} dw = \text{Re} \left( \text{Li}_2(2) - \text{Li}_2 \left( 1 - \frac{1}{u} \right) \right),$$

with  $\text{Li}_2$  the dilogarithm function. Indeed, we recall that for  $z \geq 1$ ,  $\text{Li}_2(z) = \frac{\pi^2}{6} - \int_1^z \frac{\log(t-1)}{t} dt - i\pi \log z$ . Recall that  $\text{Re}(\text{Li}_2(2)) = \frac{\pi^2}{4}$ . Using an explicit primitive of  $u \mapsto \text{Li}_2(1 + \frac{1}{u})$  (such as



$zLi_2(1-z^{-1}) + Li_2(-z) + (\log z - i\pi) \log(z+1)$ , we find that  $Re \int_0^1 Li_2(1 - \frac{1}{u}) du = \frac{\pi^2}{6}$ . Hence  $\sum_{r+\ell+s \leq n} \frac{1}{(r+\ell)(\ell+s)} \sim 2n\pi^2((1/4) - (1/6)) = \pi^2 n/6$  and so

$$\sum_{k_1+r+\ell+s \leq n} \frac{1}{(r+\ell)(\ell+s)} = \sum_{k_1=1}^{n-1} \sum_{r+\ell+s \leq n-k_1} \frac{1}{(r+\ell)(\ell+s)} = \sum_{k_1=1}^{n-1} \sum_{r+\ell+s \leq k_1} \frac{1}{(r+\ell)(\ell+s)} \sim \frac{\pi^2 n^2}{12}.$$

□

**Lemma 14.** *We have*

$$\sum_{k_1, r, \ell, s \geq 1: k_1+r+\ell+s \leq n} \frac{1}{r\ell + rs + s\ell} \sim n^2 J,$$

with

$$J := \int_{[0,1]^3} \frac{(1 - (u+v+w)) \mathbf{1}_{\{u+v+w \leq 1\}} du dv dw}{uv + uw + vw}.$$

*Proof.* We have

$$\begin{aligned} \sum_{k_1+r+\ell+s \leq n} \frac{1}{r\ell + rs + s\ell} &= \sum_{r+\ell+s \leq n} \frac{n - (r+\ell+s)}{r\ell + rs + s\ell} \\ &= n^2 \int_{[0,1]^3} f\left(\frac{[nu]}{n}, \frac{[nv]}{n}, \frac{[nw]}{n}\right) dudvdw \\ &\sim n^2 \int_{[0,1]^3} f(u, v, w) dudvdw = n^2 J, \end{aligned}$$

with  $f(u, v, w) := \frac{1-u-v-w}{uv+uw+vw} \mathbf{1}_{\{u+v+w \leq 1\}}$ , due to the Lebesgue dominated convergence theorem. □

## 5. PROOF OF THEOREM 2

**Corollary 15.** *Let  $P$  be a probability measure on  $\bar{M}$  with density  $h$  with respect to  $\bar{\mu}$ . Assume that  $h$  is in  $\mathbb{L}^2(\bar{\mu})$ . Then*

$$\mathbb{E}_P[V_n] = cn \log n + O(n).$$

*Proof of Corollary 15.* We have

$$\begin{aligned} |\mathbb{E}_P[V_n] - \mathbb{E}_{\bar{\mu}}[V_n]| &= \mathbb{E}_{\bar{\mu}}[(V_n - \mathbb{E}_{\bar{\mu}}[V_n])h] \\ &\leq \sqrt{\text{Var}_{\bar{\mu}}(V_n)} \|h\|_2 = O(n) \|h\|_2 = O(n), \end{aligned}$$

according to Theorem 3. We conclude thanks to Theorem 1. □

For any  $t > 0$ , we define  $n_t$  on  $\mathcal{M}$  by  $n_t := \max\{m \geq 0 : \sum_{k=0}^{m-1} \tau \circ T^k \leq t\}$  the number of reflections before time  $t$ .

**Corollary 16.** *Let  $h$  be a probability density with respect to  $\bar{\mu}$  belonging to  $\mathbb{L}^p(\bar{\mu})$  for some  $p > 2$ . We have*

$$\mathbb{E}_{h\bar{\mu}}[V_{n_t}] = ct \log t / \mathbb{E}_{\bar{\mu}}[\tau] + O(t), \quad \text{as } t \text{ goes to infinity.}$$

*Proof.* To simplify notations, we write  $\bar{\tau} := \mathbb{E}_{\bar{\mu}}[\tau]$ . Observe that  $n_t \leq t / \min \tau$  on  $\bar{M}$ . We define

$$D := \left| \sum_{k,j=0}^{n_t-1} \mathbf{1}_{E_{k,j}} - \sum_{k,j=0}^{\lfloor t/\bar{\tau} \rfloor - 1} \mathbf{1}_{E_{k,j}} \right| \leq 2 \sum_{k=0}^{\lfloor t/\min \tau \rfloor - 1} \sum_{j=\min(n_t, \lfloor t/\bar{\tau} \rfloor)}^{\max(n_t, \lfloor t/\bar{\tau} \rfloor) - 1} \mathbf{1}_{E_{k,j}}.$$

Due to corollary 15, it is enough to prove that  $\mathbb{E}_{h\bar{\mu}}[D] = O(t)$ . Recall that (see [15])

$$\forall m \geq 1, \exists \tilde{K}_m, \sup_{t>0} \left\| n_t - \frac{t}{\bar{\tau}} \right\|_m^m \leq \tilde{K}_m t^{\frac{m}{2}}. \quad (42)$$

Let  $\varepsilon > 0$ . Due to Proposition 10, for some  $C > 0$ , we have

$$\begin{aligned} \mathbb{E}_{h\bar{\mu}} [D \mathbf{1}_{|n_t - (t/\bar{\tau})| \leq \varepsilon t}] &\leq 2 \sum_{k=0}^{\lfloor \frac{t}{\min \tau} \rfloor - 1} \sum_{j=\lfloor \frac{t}{\bar{\tau}} - \varepsilon t \rfloor - 1}^{\lfloor \frac{t}{\bar{\tau}} + \varepsilon t \rfloor - 1} \mathbb{E}_{\bar{\mu}} [h \mathbf{1}_{E_{k,j}}] \\ &\leq 2 \sum_{k=0}^{\lfloor \frac{t}{\min \tau} \rfloor - 1} \sum_{j=\lfloor \frac{t}{\bar{\tau}} - \varepsilon t \rfloor}^{\lfloor \frac{t}{\bar{\tau}} + \varepsilon t \rfloor} \|h\|_p (\bar{\mu}(E_{k,j}))^{1-\frac{1}{p}} \\ &\leq 2 \|h\|_p \left( \lceil 2\varepsilon t \rceil + 2 \sum_{j=0}^{\lceil 2\varepsilon t \rceil} \sum_{r=1}^{\lfloor \frac{t}{\min \tau} \rfloor} C r^{\frac{1}{p}-1} \right) = O(\varepsilon t^{1+\frac{1}{p}}). \end{aligned}$$

Moreover, for any  $m \geq 1$ , we have

$$\begin{aligned} \mathbb{E}_{h\bar{\mu}} [D \mathbf{1}_{|n_t - (t/\bar{\tau})| > \varepsilon t}] &\leq \mathbb{P}(|n_t - (t/\bar{\tau})| > \varepsilon t)^{\frac{1}{2}-\frac{1}{p}} \|h\|_p \left( \mathbb{E}_{\bar{\mu}} \left[ \left( \sum_{k,j=0}^{\lfloor \frac{t}{\min \tau} \rfloor - 1} \mathbf{1}_{E_{k,j}} \right)^2 \right] \right)^{\frac{1}{2}} \\ &\leq \left( \frac{\tilde{K}_m t^{\frac{m}{2}}}{(\varepsilon t)^m} \right)^{\frac{1}{2}-\frac{1}{p}} \|h\|_p \left( \text{Var}_{\bar{\mu}}(V_{\lfloor \frac{t}{\min \tau} \rfloor}) + (\mathbb{E}_{\bar{\mu}}[V_{\lfloor \frac{t}{\min \tau} \rfloor}])^2 \right)^{\frac{1}{2}} \\ &\leq \left( \frac{\tilde{K}_m}{\varepsilon^m t^{\frac{m}{2}}} \right)^{\frac{1}{2}-\frac{1}{p}} \|h\|_p C t \log t = O\left(t \log t \varepsilon^{-\tilde{m}} t^{-\frac{\tilde{m}}{2}}\right), \end{aligned}$$

with  $\tilde{m} := m(p-2)/2p$  (due to (42), to Theorem 3 and to Theorem 1). Take  $\varepsilon = t^{-\frac{(1/p)-(\tilde{m}/2)}{\tilde{m}+1}}$ . We obtain

$$\mathbb{E}_{h\bar{\mu}}[D] = O(\varepsilon t^{1+\frac{1}{p}} \log t) = o(t),$$

by taking  $m$  large enough since  $p > 2$ . □

**Corollary 17.** *Let  $H$  be a probability density with respect to  $\nu$  on  $\mathcal{M}$  such that*

$$h : (q, \vec{v}) \mapsto \sum_{\ell \in \mathbb{Z}^2} \int_0^{\tau(q, \vec{v})} H(q + \ell + s\vec{v}, \vec{v}) ds$$

*belongs to  $\mathbb{L}^p(\bar{\mu})$  for some  $p > 2$ . Then*

$$\mathbb{E}_{H\nu}[\mathcal{V}_t] = ct \log t / \mathbb{E}_{\bar{\mu}}[\tau] + O(t), \quad \text{as } t \text{ goes to infinity.}$$

*Proof of Theorem 2.* For every  $(q, \vec{v}) \in \bar{M}$ , every  $\ell \in \mathbb{Z}^2$  and every  $s \in [0, \tau(q, \vec{v})]$ , we have

$$\mathcal{V}_t(q + \ell + s\vec{v}, \vec{v}) = O(n_t) + V_{n_t(q, \vec{v})}(q, \vec{v}) = O(t) + V_{n_t(q, \vec{v})}(q, \vec{v}).$$

So, due to (4), we have

$$\mathbb{E}_{H\nu}[\mathcal{V}_t] = O(t) + \mathbb{E}_{h\mu_{|\bar{M}}}[V_{n_t}] = \mathbb{E}_{2h \sum_i |\partial O_i| \bar{\mu}}[V_{n_t}] = ct \log t / \mathbb{E}_{\bar{\mu}}[\tau] + O(t),$$

according to Corollary 16. □

*Proof of Theorem 2.* We apply directly Corollary 17 with  $H(q, \vec{v}) = \mathbf{1}_{q \in [0,1]^2}$  and  $h(q, \vec{v}) = \tau(q, \vec{v})$ . □

## 6. ALMOST SURE CONVERGENCE

In this section we prove Corollary 4 by a classical argument (see [10] for example). Let  $\gamma \in (0, 1/2)$ . According to Theorems 1 and 3, we have

$$\text{Var}_{\bar{\mu}}(V_n)/(\mathbb{E}_{\bar{\mu}}[V_n])^2 = O(\log^{-2} n).$$

Due to the Bienaymé-Chebychev inequality and to the Borel Cantelli lemma, this implies the  $\bar{\mu}$ -almost sure convergence of  $(V_{\exp n^{\frac{1+\gamma}{2}}}/\mathbb{E}_{\bar{\mu}}[V_{\exp n^{\frac{1+\gamma}{2}}}]_n)$  to 1. Therefore, the following convergence holds  $\bar{\mu}$ -almost surely:

$$\lim_{n \rightarrow +\infty} \frac{V_{\exp n^{\frac{1+\gamma}{2}}}}{n^{\frac{1+\gamma}{2}} \exp n^{\frac{1+\gamma}{2}}} = c.$$

Now, for every integer  $N \in [\exp n^{\frac{1+\gamma}{2}}, \exp(n+1)^{\frac{1+\gamma}{2}}]$ , we have

$$\frac{V_{\exp n^{\frac{1+\gamma}{2}}}}{(n+1)^{\frac{1+\gamma}{2}} \exp(n+1)^{\frac{1+\gamma}{2}}} \leq \frac{V_N}{N \log N} \leq \frac{V_{\exp(n+1)^{\frac{1+\gamma}{2}}}}{n^{\frac{1+\gamma}{2}} \exp n^{\frac{1+\gamma}{2}}}.$$

Since  $\lim_{n \rightarrow +\infty} (n+1)^{\frac{1+\gamma}{2}} \exp(n+1)^{\frac{1+\gamma}{2}} / (n^{\frac{1+\gamma}{2}} \exp n^{\frac{1+\gamma}{2}}) = 1$ , we conclude the  $\bar{\mu}$ -almost sure convergence of  $(V_n/(n \log n))_n$  to  $c$ .

For any  $t > 0$ , we write  $n_t$  for the number of reflection times before time  $t$ . Recall that  $(t/n_t)_t$  converges  $\bar{\mu}$ -almost surely to  $\mathbb{E}_{\bar{\mu}}[\tau]$  as  $t$  goes to infinity. Hence we have,  $\bar{\mu}$ -almost surely,

$$\frac{V_{n_t}}{t \log t} \sim \frac{V_{n_t}}{\mathbb{E}_{\bar{\mu}}[\tau] n_t \log n_t} \sim \frac{c}{\mathbb{E}_{\bar{\mu}}[\tau]}, \quad \text{as } t \rightarrow +\infty.$$

Since  $V_{n_t}(q + \ell, \vec{v}) = V_{n_t}(q, \vec{v})$  for every  $(q, \vec{v}) \in \bar{M}$  and every  $\ell \in \mathbb{Z}^2$ . We also have,  $\mu$ -almost everywhere,  $\frac{V_{n_t}}{t \log t} \sim \frac{c}{\mathbb{E}_{\bar{\mu}}[\tau]}$ . Recall now that

$$\forall (q, \vec{v}) \in M, \forall s \in [0, \tau(q, \vec{v})], \quad |\mathcal{V}_t(q + s\vec{v}, \vec{v}) - V_{n_t(q, \vec{v})}(q, \vec{v})| \leq 2n_t \leq 2 \frac{t}{\min \tau}.$$

Hence, due to (4), we obtain

$$\nu \left( \left\{ \frac{\mathcal{V}_t}{t \log t} \not\rightarrow \frac{c}{\mathbb{E}_{\bar{\mu}}[\tau]} \right\} \right) \leq (\max \tau) \mu \left( \left\{ \frac{V_{n_t}}{t \log t} \not\rightarrow \frac{c}{\mathbb{E}_{\bar{\mu}}[\tau]} \right\} \right) = 0.$$

□

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